

# ON THE $K$ -THEORY OF CERTAIN EXTENSIONS OF FREE GROUPS

VASSILIS METAFTSIS AND STRATOS PRASSIDIS

## 1. INTRODUCTION

The Fibered Farrell–Jones Conjecture is the main conjecture in geometric topology. It is used for the calculation of the obstruction groups that appear in geometric rigidity and classification problems.

In this paper we are interested in the  $K$ -theory FJC and its variation  $K$ -FJCw, with finite wreath products. The  $K$ -FJCw has been proved for an extensive list of classes of groups. One notable case which remains open is the group  $\text{Aut}(F_n)$ , the automorphism group of the free group on  $n$  letters. In [12] the  $K$ -FJC is proved for  $n = 2$ . Actually, in [12], the  $K$ -FJC is proved for  $\text{Hol}(F_2)$ , the holomorph of  $F_2$ . We notice that the extension to  $K$ -FJCw is a direct computation. In this paper we extend the result to certain subgroups of  $\text{Aut}(F_n)$  that are constructed from  $\text{Hol}(F_2)$ . More precisely, there is a monomorphism  $\text{Hol}(F_n) \rightarrow \text{Aut}(F_{n+1})$ . This way we construct a sequence of groups with:

$$\mathcal{H}_{(1)} = \text{Hol}(F_2), \mathcal{H}_{(n)} = F_{n+1} \rtimes \mathcal{H}_{(n-1)}, n \geq 2.$$

Notice that  $\mathcal{H}_{(n)} < \text{Hol}(F_{n+1})$ . The main theorem of the paper is the following.

**Theorem** (Main Theorem). *The  $K$ -FJCw holds for the groups  $\mathcal{H}_{(n)}$ .*

As an application of the Main Theorem, we calculate the lower  $K$ -theory groups of  $\mathcal{H}_{(n)}$ :

- (1)  $K_i(\mathbb{Z}\mathcal{H}_{(n)}) = 0, i \leq -1$ .
- (2)  $\tilde{K}_0(\mathbb{Z}\mathcal{H}_{(n)}) \cong NK_0(\mathbb{Z}D_4) \oplus NK_0(\mathbb{Z}D_4)$ .
- (3)  $Wh(\mathcal{H}_{(n)}) \cong NK_1(\mathbb{Z}D_4) \oplus NK_1(\mathbb{Z}D_4)$ .

The main point of the general FJCw is that the  $K$ -theory of a group can be computed from the  $K$ -theory of its virtually cyclic subgroups. These are of three types: finite groups, groups that admit an epimorphism to  $\mathbb{Z}$  with finite kernel, groups that admit an epimorphism to  $D_\infty$  (the infinite dihedral group) with finite kernel. In [6], it was shown that the first two classes are enough for the  $K$ -FJCw. That means that the  $K$ -groups of a group can be computed from the  $K$ -theory of finite subgroups and groups of the form  $F \rtimes \mathbb{Z}$  with  $F$  finite. For the proof of the main theorem, we use two properties of the FJCw:

- (1) If the FJCw holds for a group  $G$ , it holds for all the subgroups of  $G$ .

(2) Let

$$1 \rightarrow H \xrightarrow{f} G \xrightarrow{g} K \rightarrow 1$$

be an exact sequence of groups. We assume that:

- (a) The FJCw holds for  $K$ .
- (b) The FJCw holds for  $g^{-1}(V)$  where  $V$  is a virtually cyclic subgroup of  $K$ .

Then the FJCw holds for  $G$

In [12], it was shown that the finite subgroups of  $\text{Hol}(F_2)$  are isomorphic to one of the groups of the following list:

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, D_2, D_4.$$

The only subgroups of the second type are isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ . That implies that the finite subgroups of  $\mathcal{H}_{(n)}$  are isomorphic to one of the above list. Also, we show that the subgroups of the second type are isomorphic to products of finite groups times  $\mathbb{Z}$ . In other words, semidirect products do not appear.

For the proof of the main theorem, we use induction and Property (2) of the FJCw. We show that the groups that are the inverses images of virtually cyclic groups of first or second type are either hyperbolic groups or CAT(0)-groups for which the  $K$ -FJCw holds.

For the calculations of the lower  $K$ -groups, we notice that the part of  $K$ -theory of  $\mathcal{H}_{(n)}$  that is detected from the finite groups vanishes. The result follows from the calculation of the cockerel of the map from the  $K$ -theory detected from the finite subgroups to the total  $K$ -theory. For this, we use [1].

## 2. PRELIMINARIES AND NOTATION

For a group  $G$  let  $\text{Aut}(G)$  the group of automorphisms of  $G$  and  $\text{Hol}(G)$  the universal split extension determined by  $G$ :

$$1 \rightarrow G \rightarrow \text{Hol}(G) \rightarrow \text{Aut}(G) \rightarrow 1.$$

In general, there is an embedding  $E : \text{Hol}(G) \rightarrow \text{Aut}(G * \mathbb{Z})$  given by: for  $g \in G$ ,

$$E(g)(x) = \begin{cases} x, & \text{for } x \in G \\ gxg^{-1}, & \text{for } x \in \mathbb{Z} \end{cases}$$

and for  $\alpha \in \text{Aut}G$ ,

$$E(\alpha)(x) = \begin{cases} \alpha(x), & \text{for } x \in G \\ x, & \text{for } x \in \mathbb{Z} \end{cases}$$

Thus, we can define the split group extension  $(G * \mathbb{Z}) \rtimes (\text{Hol}(G)) < \text{Hol}(G * \mathbb{Z}) < \text{Aut}((G * \mathbb{Z}) * \mathbb{Z})$ .

Inductively, we define  $\mathcal{H}_{(i)}(G)$  to be:

$$\mathcal{H}_{(0)}(G) = G, \mathcal{H}_{(1)}(G) = \text{Hol}(G), \mathcal{H}_{(n)}(G) = (G * F_{n-1}) \rtimes (\mathcal{H}_{(n-1)}(G)), \quad n \geq 2.$$

where  $F_{n+1}$  is the group on  $(n+1)$ -generators. We write  $\mathcal{H}_{(n)} = \mathcal{H}_{(n)}(F_2)$ , for the group corresponding to  $F_2$ . Then there is a split exact sequence:

$$1 \rightarrow F_{n+1} \rightarrow \mathcal{H}_{(n)} \rightarrow E(\mathcal{H}_{(n-1)}) \rightarrow 1.$$

We are interested in the Fibered Farrell-Jones Conjecture (FJC) for the groups  $\mathcal{H}_{(n)}$ . We will review the general constructions. Let  $G$  be a group and  $\mathcal{C}$  be a class of subgroups. Then  $E_{\mathcal{C}}G$  denotes the classifying space of the class  $\mathcal{C}$ . We are interested in the following classes of subgroups of  $G$ :

- 1, the class of the trivial subgroup.
- $\mathcal{F}$ , the class of finite subgroups.
- $\mathcal{FBC}$ , the class of finite-by-cyclic subgroups.
- $\mathcal{VC}$ , the class of virtually cyclic subgroups.
- $\mathcal{All}$ , the class of all subgroups.

It is obvious that  $1 \subset \mathcal{F} \subset \mathcal{FBC} \subset \mathcal{VC} \subset \mathcal{All}$ . Instead of the classical  $K$ -theoretic FJC, we will consider the  $K$ -theoretic Isomorphic Conjecture with coefficients in an additive category  $\mathcal{A}$ . It is known that this implies also the Fibered Isomorphism Conjecture ([2]). It states that the assembly map

$$H_n^G(E_{\mathcal{VC}}G; \mathbf{K}_{\mathcal{A}}) \rightarrow H_n^G(E_{\mathcal{All}}G; \mathbf{K}_{\mathcal{A}}) = H_n^G(pt; \mathbf{K}_{\mathcal{A}})$$

is an isomorphism. If a group satisfies the Conjecture, we say that the group satisfies the FJC. We say that a group  $G$  satisfies the FJCw if the wreath product  $G \wr H$  satisfies the FJC for each finite group  $H$  and with coefficients.

We need the following basic facts:

*Remark 2.1.* (1) Word hyperbolic groups satisfy the  $K$ -FJCw ([3]).

(2) CAT(0)-groups satisfy the  $K$ -FJCw ([17]).

(3) Strongly poly-free groups or, more generally, weak strongly poly-surface groups satisfy the  $K$ -FJCw ([14]).

(4) Let  $G$  satisfies the  $K$ -FJCw and

$$1 \rightarrow G \rightarrow K \rightarrow H \rightarrow 1$$

be an exact sequence with  $H$  finite. Then  $K$  satisfies the  $K$ -FJCw ([14]).

In [1] it was shown that, for a ring  $R$ , the relative map

$$H_n^G(E_{\mathcal{F}}G; \mathbf{K}R^{-\infty}) \rightarrow H_n^G(E_{\mathcal{VC}}G; \mathbf{K}R^{-\infty})$$

is a split injection. Also, in [6] it was shown that the natural map

$$H_n^G(E_{\mathcal{FBC}}G; \mathbf{K}R^{-\infty}) \rightarrow H_n^G(E_{\mathcal{VC}}G; \mathbf{K}R^{-\infty})$$

is an isomorphism. Taking the corresponding cockerels, we have that

$$H_n^G(E_{\mathcal{F}}G \rightarrow E_{\mathcal{V}\mathcal{C}}G; \mathbf{K}R^{-\infty}) \cong H_n^G(E_{\mathcal{F}}G \rightarrow E_{\mathcal{F}\mathcal{B}\mathcal{C}}G; \mathbf{K}R^{-\infty})$$

Now, let the group  $G$  satisfy the condition  $\mathcal{M}_{\mathcal{F}\subset\mathcal{F}\mathcal{B}\mathcal{C}}$  of [9], which states that every infinite group in  $\mathcal{F}\mathcal{B}\mathcal{C}$  is contained in a unique maximal group in  $\mathcal{F}\mathcal{B}\mathcal{C}$ . Then

$$\bigoplus_{V \in \mathcal{M}} H_n^{N_G(V)}(E_{\mathcal{F}}N_G(V) \rightarrow E_1W_G(V); \mathbf{K}R^{-\infty}) \xrightarrow{\cong} H_n^G(E_{\mathcal{F}}G \rightarrow E_{\mathcal{F}\mathcal{B}\mathcal{C}}G; \mathbf{K}R^{-\infty}).$$

Here  $\mathcal{M}$  is a set of representatives of the conjugacy classes of the maximal infinite groups in  $\mathcal{F}\mathcal{B}\mathcal{C}$  and  $W_G(V) = N_G(V)/V$  is the Weyl group of  $V$  ([9, Corollary 6.1]) Remark 6.2 in [9] implies that there is a spectral sequence

$$(1) \quad \begin{aligned} E_{p,q}^2 &= H_p^{W_G(V)}(E_1W_G(V); H_q^V(E_{\mathcal{F}}N_G(V) \rightarrow \{pt\}); \mathbf{K}R^{-\infty}) \Rightarrow \\ &H_{p+q}^{N_G(V)}(E_{\mathcal{F}}N_G(V) \rightarrow E_1W_G(V); \mathbf{K}R^{-\infty}) \end{aligned}$$

That is obtained by choosing  $X = E_1W_G(V)$  and noticing that  $E_{\mathcal{F}}N_G(V) \times E_1W_G(V)$  is a space of type  $E_{\mathcal{F}}N_G(V)$  with the diagonal action. Also, Example 6.3 in [9] implies that, if  $V = F \rtimes \mathbb{Z}$  then  $H_q^V(E_{\mathcal{F}}N_G(V) \rightarrow \{pt\})$  is the non-connective version of Farrell's twisted Nil-term. Thus the spectral sequence becomes:

$$(2) \quad E_{p,q}^2 = H_p^{W_G(V)}(E_1W_G(V); \mathbf{Nil}_R) \Rightarrow H_{p+q}^{N_G(V)}(E_{\mathcal{F}}N_G(V) \rightarrow E_1W_G(V); \mathbf{K}R^{-\infty})$$

*Remark 2.2.* In [12], it was shown that:

- (1) The finite subgroups of  $\text{Aut}(F_2)$  and  $\text{Hol}(F_2)$  are isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z}$ ,  $\mathbb{Z}/4\mathbb{Z}$ ,  $D_2$  and  $D_4$ . The maximal finite subgroups are  $\mathbb{Z}/3\mathbb{Z}$  and  $D_4$ . From the construction, the same is true for  $\mathcal{H}_{(n)}$  for all  $n \geq 1$ .
- (2) Although this is not explicitly shown in [12], up to isomorphism, there are various infinite  $\mathcal{F}\mathcal{B}\mathcal{C}$  subgroups of  $\text{Hol}(F_2)$  which are isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ . In fact, the construction of  $\mathcal{H}_{(n)}$  shows that the infinite  $\mathcal{F}\mathcal{B}\mathcal{C}$  subgroups are  $F \times \mathbb{Z}$  where  $F < \mathcal{H}_{(n)}$  is finite. The subgroup  $\mathbb{Z}$  is a subgroup in the factors that are complementary to  $\text{Hol}(F_2)$ . That is because each element that belongs to  $F_i < \mathcal{H}_{(n)}$ ,  $3 \leq i \leq n$ , commutes with the subgroups of  $\text{Hol}(F_2)$ .

Thus the maximal infinite  $\mathcal{F}\mathcal{B}\mathcal{C}$  subgroups are of the following types:  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ ,  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}$  and  $D_4 \times \mathbb{Z}$ .

We will show that certain mapping tori of the free groups that are contained in  $\mathcal{H}_{(n)}$  are  $\text{CAT}(0)$ -groups. Notice that, by the work of Brady [4], all mapping tori  $F_2 \rtimes \mathbb{Z}$  contained in  $F_2 \rtimes \text{Aut}(F_2)$  are  $\text{CAT}(0)$ . We show that this is true for all mapping tori  $F_{n+1} \rtimes \mathbb{Z}$  contained in  $F_{n+1} \rtimes E(\mathcal{H}_{(n-1)})$ .

**Proposition 2.3.** *Let  $G_n = F_{n+1} \rtimes \mathbb{Z} < F_{n+1} \rtimes E(\mathcal{H}_{(n-1)})$ . Then  $G_n$  is  $\text{CAT}(0)$ .*

*Proof.* Set  $G_n = F_{n+1} \rtimes \mathbb{Z} < \mathcal{H}_{(n)}$ . By definition, the  $\mathbb{Z}$ -action on  $F_{n+1}$  is from an element of  $\mathcal{H}_{(n-1)}$ . That means that in the first two generators it is an automorphism of the free group they

generate and on the other generators is conjugation by words on the previous generators. We will use induction. For  $n = 1$ ,  $G_1 = F_2 \rtimes \mathbb{Z}$  which is  $\text{CAT}(0)$  ([4]).

For general  $n$ , let  $F_{n+1} = \langle x_1, x_2, \dots, x_{n+1} \rangle$ . Notice that  $\mathcal{H}_{(n-1)} = F_n \rtimes \mathcal{H}_{(n-2)}$ . Then every  $g \in \mathcal{H}_{(n-1)}$  can be written as  $g = g_1 g_2$ , with  $g_1 \in F_n$  and  $g_2 \in \mathcal{H}_{(n-2)}$ . Then the embedding  $\mathcal{H}_{(n-1)}$  in  $\text{Aut}(F_{n+1})$  sends  $g$  to  $\tilde{g}$  with:

$$\tilde{g}(x_i) = g_2(x_i), \quad i = 1, 2, \dots, n, \quad \tilde{g}(x_{n+1}) = g_1 x_{n+1} g_1^{-1}.$$

Then

$$G_n = F_{n+1} \rtimes_{\tilde{g}} \mathbb{Z} = \langle t, x_1, x_2, \dots, x_{n+1} : tx_i t^{-1} = g_2(x_i), \quad i = 1, 2, \dots, n, \quad tx_{n+1} t^{-1} = g_1 x_{n+1} g_1^{-1} \rangle$$

with  $g_1$  a word in  $x_i$ ,  $i = 1, 2, \dots, n$ . Setting  $\alpha = g_1^{-1} t$  and solving for  $t$ , we get

$$G_n = \langle \alpha, x_1, x_2, \dots, x_{n+1} : \alpha x_i \alpha^{-1} = g_1^{-1} g_2(x_i) g_1, \quad i = 1, 2, \dots, n \rangle *_{\mathbb{Z}} \langle \alpha, x_{n+1} : [\alpha, x_{n+1}] = 1 \rangle,$$

where  $\mathbb{Z} = \langle \alpha \rangle$ . Then  $G_n = H *_{\mathbb{Z}} \mathbb{Z}^2$ . To characterize the group  $H$ , we set  $\beta = g_1 \alpha$ ,

$$H = \langle \beta, x_1, x_2, \dots, x_n : \beta x_i \beta^{-1} = g_2(x_i), \quad i = 1, 2, \dots, n \rangle.$$

If  $n = 2$ , then  $H = F_2 \rtimes \mathbb{Z} = G_1$ . If  $n > 2$ , then  $g_2 \in \mathcal{H}_{(n-2)}$ . Thus  $H = G_{n-1}$ . So  $G_n = G_{n-1} *_{\mathbb{Z}} \mathbb{Z}^2$  which is  $\text{CAT}(0)$  by induction and [5], Part II, Proposition 11.19.  $\square$

The following is in [17].

**Corollary 2.4.** *The groups  $G_n$  in Proposition 2.3 satisfy the  $K$ -FJCw.*

In [12], it was shown that the only infinite, virtually cyclic subgroup of Type (I) in  $\text{Hol}(F_2)$  is  $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

**Corollary 2.5.** *The group  $F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) < \mathcal{H}_{(n-1)}$  satisfies the  $K$ -FICw.*

*Proof.* In [16], it was shown that  $\text{CAT}(0)$ -groups satisfy the  $K$ -FJCw. That means that finite extensions of  $\text{CAT}(0)$ -groups satisfy the  $K$ -FJCw. The result follows.  $\square$

*Remark 2.6.* In the Appendix, we will show that certain groups that appear in Corollary 2.5 are  $\text{CAT}(0)$ .

We are also able to prove the following.

**Proposition 2.7.** *Let  $D$  be a finite subgroup of  $\text{Aut}(F_2)$  and  $E(D)$  its image in  $\mathcal{H}_{(n)}$ . Then there are infinite cyclic-by-finite subgroups of the form  $\mathbb{Z} \times E(D)$  in  $\text{Hol}(F_n) \setminus \text{Aut}(F_n)$ ,  $n \geq 3$ . Moreover, every  $G = F_m \rtimes (\mathbb{Z} \times E(D))$  in  $\mathcal{H}_{(m-2)}$  is  $\text{CAT}(0)$  for all  $m > n$ .*

*Proof.* Assume that  $F_n = \langle x_1, \dots, x_n \rangle$ . By definition of  $E$ , for all  $\phi \in E(D)$  we have that  $\phi(x_i) = x_i$  for every  $i > 2$ . Hence, in  $\text{Hol}(F_n)$ ,  $\langle x_i \rangle$  with  $i > 2$  commutes with  $\phi$  for all  $\phi \in E(D)$  and so  $\langle x_i, E(D) \rangle$  is isomorphic to  $\mathbb{Z} \times E(D)$  for every  $i > 2$ .

Let us now see  $\mathbb{Z} \times E(D)$  as a subgroup of  $\text{Hol}(F_n)$ ,  $\mathbb{Z} = \langle x_n \rangle$ ,  $n \geq 3$  and embed it in  $\text{Aut}(F_{n+1})$ . Then the action of  $\mathbb{Z} = \langle \xi_{x_n} \rangle$  on  $x_{n+1}$  is conjugation by  $x_n$  and is trivial on every other generator of  $F_{n+1}$ . Then  $G$  has a presentation of the following form

$$G = \langle x_1, \dots, x_{n+1}, \xi_{x_n}, E(D) \mid [\xi_{x_n}, E(D)] = 1, [\xi_{x_n}, x_j] = 1 \quad \forall j < n+1, \\ \xi_{x_n} x_{n+1} \xi_{x_n}^{-1} = x_n x_{n+1} x_n^{-1}, [x_i, E(D)] = 1 \quad \forall i > 2 \rangle.$$

Set  $z = x_n^{-1} \xi_{x_n}$ , and get rid of  $x_n$  to get the presentation

$$G = \langle x_1, \dots, x_{n-1}, z, x_{n+1}, E(D) \mid [\xi_{x_n}, E(D)] = 1, [z, x_j] = 1 \quad \forall j < n+1, j \neq n \\ [\xi_{x_n}, z] = 1, [x_i, E(D)] = 1 \quad \forall i > 2, i \neq n, [z, E(D)] = 1 \rangle.$$

Now decompose  $G$  as an amalgamated free product  $G = G_1 *_{\mathbb{Z} \times E(D)} G_2$  with

$$G_1 = \langle x_1, x_2, \xi_{x_n}, E(D) \rangle \cong F_2 \rtimes (\mathbb{Z} \times E(D)),$$

$$G_2 = \langle x_3, \dots, x_{n-1}, x_{n+1}, z, \xi_{x_n}, E(D) \rangle \cong \langle x_3, \dots, x_{n-1}, x_{n+1}, z, \xi_{x_n} \rangle \times E(D).$$

Now notice that the subgroup generated by  $\langle x_3, \dots, x_{n-1}, x_{n+1}, z, \xi_{x_n} \rangle$  has a presentation

$$\langle x_3, \dots, x_{n-1}, x_{n+1}, z, \xi_{x_n} \mid [\xi_{x_n}, z] = 1, [\xi_{x_n}, x_j] = 1 \quad \forall j < n, [z, x_{n+1}] = 1 \rangle$$

which makes it a right-angled Artin group and so is  $\text{CAT}(0)$  by [7]. Thus  $G$  is  $\text{CAT}(0)$  by [5], Part II, Proposition 11.19.  $\square$

*Remark 2.8.* In the last proposition, we showed that the group  $G$  is  $\text{CAT}(0)$  and thus it satisfies the  $K$ -FJCw. But we can use Proposition 2.3 to show directly that  $G$  satisfies the  $K$ -FJCw. That is done as in Corollary 2.5.

*Remark 2.9.* Note the following two properties of the groups described in Lemma 2.7.

- (1) Every such subgroup is contained into a maximal cyclic by finite subgroup. This is an immediate consequence of the fact that  $\text{Aut}(F_2)$  decomposes as an amalgamated free product with maximal elements of finite order.
- (2) The normalizer of every maximal such subgroup coincides with the normalizer of its finite subgroup in  $\text{Aut}(F_2)$ .

Now let us introduce some notation from [12]. The group  $\text{Aut}(F_2)$  admits a presentation of the form

$$\langle p, x, y, \tau_a, \tau_b \mid x^4 = p^2 = (px)^2 = 1, (py)^2 = \tau_b, x^2 = y^3 \tau_b^{-1} \tau_a, \\ p^{-1} \tau_a p = x^{-1} \tau_a x = y^{-1} \tau_a y = \tau_b, p^{-1} \tau_b p = \tau_a, x^{-1} \tau_b x = \tau_a^{-1}, y^{-1} \tau_b y = \tau_a^{-1} \tau_b \rangle$$

where  $\tau_a, \tau_b$  are the inner automorphism of  $F_2$  corresponding to  $a, b$  respectively. Moreover, any element of  $\text{Aut}(F_2)$  can be written uniquely in the form  $p^r u(x, y) x^{2s} w(\tau_a, \tau_b)$  where  $r, s \in \{0, 1\}$ ,  $w(\tau_a, \tau_b)$  is a reduced word in  $\text{Inn}(F_2)$  and  $u(x, y)$  is a reduced word where  $x, y, y^{-1}$  are the only powers of  $x, y$  appearing (see [11, 10]).

Moreover, a presentation for  $\mathrm{GL}_2(\mathbb{Z})$  is given by

$$\mathrm{GL}_2(\mathbb{Z}) = \langle P, X, Y \mid X^4 = P^2 = (PX)^2 = (PY)^2 = 1, X^2 = Y^3 \rangle$$

and  $\mathrm{Aut}(F_2)$  maps homomorphically onto  $\mathrm{GL}_2(\mathbb{Z})$  by  $p \mapsto P, x \mapsto X, y \mapsto Y, \tau_a, \tau_b \mapsto 1$ .

**Lemma 2.10.** *Let  $D_4$  be the subgroup of  $\mathrm{Aut}(F_2)$  generated by  $\langle p, x \rangle$ . Then the normalizer  $N_{\mathrm{Aut}(F_2)}(D_4)$  of  $D_4$  in  $\mathrm{Aut}(F_2)$  is  $D_4$  itself.*

*Proof.* Let  $p^r u(x, y) x^{2s} w(\tau_a, \tau_b)$  be an element of  $\mathrm{Aut}(F_2)$  that belongs to the normalizer of  $D_4$ . Then it necessarily conjugates elements of order 4 to elements of order 4. But the only elements of order 4 in  $\mathrm{Aut}(F_2)$  are conjugates of  $x^{\pm 1}$  (see [11]). Hence we have the following relation

$$(3) \quad p^r u x^{2s} w \cdot x \cdot w^{-1} x^{-2s} u^{-1} p^{-r} = x^{\pm 1}$$

or equivalently

$$u x^{2s} w \cdot x \cdot w^{-1} x^{-2s} u^{-1} = p^r x^{\pm 1} x^{-r}$$

i.e.

$$u x^{2s} w \cdot x \cdot w^{-1} x^{-2s} u^{-1} = x^{\pm 1}.$$

Now project this relation to  $\mathrm{GL}_2(\mathbb{Z})$ . It reduces to  $U(X, Y) X^{2s} X X^{-2s} U^{-1} = X^{\pm 1}$  or  $U X U^{-1} = X^{\pm 1}$ . This last relation implies that  $U = X$  and therefore  $u = x$  since the projection maps  $y$  to  $D_6 \setminus D_2$  and  $x$  to  $D_4 \setminus D_2$ , and therefore  $U$  and  $X$  freely generate a free group.

Thus (3) reduces to  $w(\tau_a, \tau_b) x w^{-1}(\tau_a, \tau_b) = x^{\pm 1}$  which implies that  $w = 1$ . Hence the only words of  $\mathrm{Aut}(F_2)$  that normalize  $x$  are of the form  $p^r x^s$ , hence  $N_{\mathrm{Aut}(F_2)}(D_4) = D_4$ .  $\square$

**Corollary 2.11.** *The normalizer  $N_{\mathcal{H}_{(n)}}(D_4 \times \mathbb{Z}) = D_4 \times \mathbb{Z}$ .*

*Proof.* It follows from 2.10 and Remark 2.9.  $\square$

### 3. THE LOWER $K$ -THEORY FOR $\mathcal{H}_{(n)}$

The algebraic calculations of the previous section allows us to prove the  $K$ -theoretic Isomorphism Conjecture for the groups  $\mathcal{H}_{(n)}$ , using induction.

We start with the first case.

**Proposition 3.1.** *The groups  $\mathrm{Aut}(F_2)$  and  $\mathrm{Hol}(F_2)$  satisfy the  $K$ -FJCw.*

*Proof.* In [12], it was shown that the groups  $\mathrm{Aut}(F_2)$  and  $\mathrm{Hol}(F_2)$  are strongly poly-free groups. The result follows from [14].  $\square$

**Theorem 3.2.** *The groups  $\mathcal{H}_{(n)}$  satisfy the  $K$ -FJCw.*

*Proof.* We will use induction on  $n$ . For  $n = 0$ ,  $\mathcal{H}_{(0)} = F_2$ , for which the fibered isomorphism conjecture holds ([3]). For  $n = 1$ , the result in Proposition 3.1. So we assume that  $n > 1$ . Then we have an exact sequence

$$1 \rightarrow F_{n+1} \rightarrow \mathcal{H}_{(n)} \xrightarrow{p} \mathcal{H}_{(n-1)} \rightarrow 1.$$

We assume that the  $K$ -FJCw holds for  $\mathcal{H}_{(n-1)}$ . So, it suffices to show that the  $K$ -FJCw holds for  $p^{-1}(V)$  where  $V$  is in  $\mathcal{FBC}$  of  $\mathcal{H}_{(n-1)}$ . There are two cases:

- (1) Let  $V < \mathcal{H}_{(n-1)}$  be finite. Then  $p^{-1}(V) \cong F_{n+1} \rtimes V$  is a hyperbolic group. Thus it satisfies the  $K$ -FJCw ([3], [16]).
- (2) Let  $V$  is an infinite  $\mathcal{FBC}$  subgroup of  $\mathcal{H}_{(n-1)}$ . There are three cases:
  - (a) Let  $V \cong \mathbb{Z}$ . Proposition 2.3 implies that  $p^{-1}(V)$  is  $\text{CAT}(0)$ . Thus it satisfies the  $K$ -FJCw (Corollary 2.4).
  - (b) Let  $V \cong \mathbb{Z} \times \mathbb{Z} < \text{Hol}(F_2)$ . The result follows from Corollary 2.5.
  - (c) Let  $V \cong D \times \mathbb{Z}$  where  $D$  is finite and  $D \times \mathbb{Z} \in \mathcal{H}_{(n-1)} \setminus \text{Hol}(F_2)$ . The Proposition 2.7 implies that  $p^{-1}(V)$  is  $\text{CAT}(0)$ . Thus it satisfies the  $K$ -FJCw.

The result follows from Remark 2.1. □

Using Theorem 3.2, we calculate the lower  $K$ -theory of  $\mathcal{H}_{(n)}$ .

**Theorem 3.3.** *The groups  $K_q(\mathbb{Z}\mathcal{H}_{(n)}) = 0$ , for  $q \leq -1$ . For the reduced  $K$ -groups in the other dimensions, we have, for  $q = 0, 1$ ,*

$$\tilde{K}_q(\mathbb{Z}\mathcal{H}_{(n)}) = \begin{cases} 0, & n = 0, 1 \\ NK_q(\mathbb{Z}D_4) \oplus NK_q(\mathbb{Z}D_4), & n \geq 2 \end{cases}$$

*Proof.* We assume  $n \geq 1$ , because  $\mathcal{H}_{(0)} = F_2$ , which is a hyperbolic group, and  $\mathcal{H}_{(0)} = \text{Hol}(F_2)$  and the result was proved in [12]. Since the groups  $\mathcal{H}_{(n)}$  satisfy the  $K$ -FJCw we have that

$$K_q(\mathbb{Z}\mathcal{H}_{(n)}) \cong H_q^{\mathcal{H}_{(n)}}(E_{\mathcal{FBC}}\mathcal{H}_{(n)}; \mathbf{K}\mathbb{Z}^{-\infty}) \cong H_q^{\mathcal{H}_{(n)}}(E_{\mathcal{F}}\mathcal{H}_{(n)}; \mathbf{K}\mathbb{Z}^{-\infty}) \oplus \bigoplus_{V \in \mathcal{M}} H_q^{N_{\mathcal{H}_{(n)}}V}(E_{\mathcal{F}}N_{\mathcal{H}_{(n)}}V \rightarrow E_1W_{\mathcal{H}_{(n)}}V; \mathbf{K}\mathbb{Z}^{-\infty}),$$

where  $\mathcal{M}$  is a set of representatives of the conjugacy classes of maximal infinite groups in  $\mathcal{FBC}$ .

The calculations of [12] show that  $H_n^{\mathcal{H}_{(n)}}(E_{\mathcal{F}}\mathcal{H}_{(n)}; \mathbf{K}\mathbb{Z}^{-\infty}) = 0$ , for  $n \leq 1$ . For each of the summands, there is a spectral sequence

$$E_{p,q}^2 = H_p^{W_{\mathcal{H}_{(n)}}V}(E_1W_{\mathcal{H}_{(n)}}V; H_q^V(E_{\mathcal{F}}V \rightarrow pt; \mathbf{K}\mathbb{Z}^{-\infty})) \Rightarrow H_{p+q}^{N_{\mathcal{H}_{(n)}}V}(E_{\mathcal{F}}N_{\mathcal{H}_{(n)}}V \rightarrow E_1W_{\mathcal{H}_{(n)}}V; \mathbf{K}\mathbb{Z}^{-\infty})$$

In Remark 2.2 it was shown that the maximal infinite groups in  $\mathcal{H}_{(n)}$  are of the following types  $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$ ,  $(\mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}$ ,  $D_4 \times \mathbb{Z}$  and there is only one conjugacy class for each of the groups  $\mathbb{Z}_3 \times \mathbb{Z}$  and  $D_4 \times \mathbb{Z}$ .



- (1) When  $V$  is  $\mathbb{Z}_2 \times \mathbb{Z}$  or  $\mathbb{Z}_3 \times \mathbb{Z}$ , then  $H_q^V(E_{\mathcal{F}}V \rightarrow pt; \mathbf{K}\mathbb{Z}) = 0$ ,  $q \leq 1$  because the Nil-groups of the two cyclic groups vanish. Thus, for these groups, (Spectral sequence (2))

$$H_n^{N_{\mathcal{H}(n)}V}(E_{\mathcal{F}}N_{\mathcal{H}(n)}V \rightarrow E_1W_{\mathcal{H}(n)}V; \mathbf{K}\mathbb{Z}^{-\infty}) = 0, \quad i \leq 1$$

- (2) For  $V = D_4 \times \mathbb{Z}$ , we have that  $N_{\mathcal{H}(n)}V = V$  (Corollary 2.11) and the spectral sequence (Spectral sequence (1)) reduces to the isomorphism for  $q \leq 1$

$$H_q^{N_{\mathcal{H}(n)}V}(E_{\mathcal{F}}N_{\mathcal{H}(n)}V \rightarrow E_1W_{\mathcal{H}(n)}V; \mathbf{K}\mathbb{Z}^{-\infty}) \cong H_q^V(E_{\mathcal{F}}V \rightarrow pt; \mathbf{K}\mathbb{Z}^{-\infty}) \cong NK_q(\mathbb{Z}D_4) \oplus NK_q(\mathbb{Z}D_4).$$

It is known that  $NK_q(\mathbb{Z}D_4) = 0$  for  $q \leq -1$  and it is infinitely generated for  $q = 0, 1$  ([18]).

Combining the above information, we have that ( $n \geq 2$ ):

- (1)  $K_i(\mathbb{Z}\mathcal{H}(n)) = 0$ ,  $i \leq -1$ .
- (2)  $\tilde{K}_0(\mathbb{Z}\mathcal{H}(n)) \cong NK_0(\mathbb{Z}D_4) \oplus NK_0(\mathbb{Z}D_4)$ .
- (3)  $Wh(\mathcal{H}(n)) \cong NK_1(\mathbb{Z}D_4) \oplus NK_1(\mathbb{Z}D_4)$ .

□

*Remark 3.4.* In [18] it was shown that  $NK_0(\mathbb{Z}D_4)$  is isomorphic to the direct sum of infinite free  $\mathbb{Z}_2$ -module with a countably infinite free  $\mathbb{Z}_4$ -module. Also,  $NK_1(\mathbb{Z}D_4)$  is a countably infinite torsion group of exponent 2 or 4.

#### 4. CONCLUDING REMARKS

In general, if the group  $G$  is linear (i.e. it admits a faithful finite dimensional real or complex representation) then the  $K$ -FJCw can be proved for  $G$ . The problem is that  $\text{Aut}(F_n)$  is not linear for  $n \geq 3$ . The group that was used to show that  $\text{Aut}(F_n)$  is not linear is the Formanek-Procesi group,  $FP$  ([8]). That group given by a split extension:

$$1 \rightarrow F_3 \xrightarrow{f} FP \xrightarrow{p} F_2 \rightarrow 1$$

and has presentation:

$$FP = \langle \alpha_1, \alpha_2, \alpha_3, \phi_1, \phi_2 : \phi_i \alpha_j \phi_i^{-1} = \alpha_j, \phi_i \alpha_3 \phi_i^{-1} = \alpha_3 \alpha_i, i, j = 1, 2 \rangle.$$

In [8], it was shown that  $FP$  is not linear and  $FP < \text{Aut}(F_3)$ . On the other hand, it is obvious that the group  $G$  is not word hyperbolic (since it contains a subgroup isomorphic to  $\mathbb{Z} \times \mathbb{Z}$ ) and not known if it is  $\text{CAT}(0)$ . The point here is that  $FP$  has “enough”  $\text{CAT}(0)$ -subgroups so that it satisfies the  $K$ -FJCw.

**Proposition 4.1.** *The group  $FP$  satisfies the  $K$ -FJCw.*

*Proof.* All the virtually cyclic subgroups of  $F_2$  are infinite cyclic. Let  $V$  be such a group that is generated by a word on  $\phi_1$  and  $\phi_2$  and their inverses. Then the action of the generator on  $F_3$  fixes the first two generators and sends  $\alpha_3$  to the element  $\alpha_3 c$  where  $c$  is a word in  $\alpha_1, \alpha_2$  and their

inverses. Then  $F_3 \rtimes V$  is a CAT(0)-group from Theorem 4.4 in [15]. Thus, it satisfies the  $K$ -FJCw. Therefore  $FP$  satisfies the  $K$ -FJCw.  $\square$

## 5. APPENDIX

We will show the result that was stated in Remark 2.6, that certain groups of type  $F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) < \mathcal{H}_{(n-1)}$  are CAT(0).

Our investigation, similar to the one in [12, Proposition 3.2] shows that subgroups of  $\text{Aut}(F_2)$  isomorphic to  $F_2 \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$  occur when the action of  $\mathbb{Z}/2\mathbb{Z}$  on  $F_2$  is of the following form

$$t(x_1) = x_1^{-1}, \quad t(x_2) = x_2 \quad \text{or} \quad t(x_1) = x_1, \quad t(x_2) = x_2^{-1}.$$

We show that in all the above cases these subgroups are CAT(0).

**Lemma 5.1.** *Let  $\mathbb{Z}/2\mathbb{Z} \cong \langle t_1 \rangle < \text{Aut}(F_2)$  be such that  $t_1(x_1) = x_1^{-1}$  and  $t_1(x_2) = x_2$ , as above. Let  $\mathbb{Z} \cong \langle t_2 \rangle < \text{Hol}(F_2)$  so that  $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} < \text{Hol}(F_2)$ . Then we have the following cases:*

- (1) *If  $t_2 \notin \text{Aut}(F_2)$ , then  $t_2 = x_2^k \in F_2$ ,  $k \in \mathbb{Z}$ ,  $k \neq 0$ .*
- (2) *If  $t_2 \in \text{Aut}(F_2)$ , then we have  $t_2(x_1) = x_2^k x_1^{\pm 1} x_2^{-k}$ ,  $t_2(x_2) = x_2^{\pm 1}$ ,  $k \neq 0$  (four cases).*

*Proof.* First, we assume  $t_2 \notin \text{Aut}(F_2)$ . Because  $t_1$  and  $t_2$  commute,  $t_1$  acts trivially on  $t_2$ . Let  $w(x_1, x_2)$  be the word in  $F_2$  representing  $t_2$ . Then,  $w(x_1^{-1}, x_2) = w(x_1, x_2)$ . That means that  $x_1$  does not appear in  $w(x_1, x_2)$ . Thus  $w(x_1, x_2) = x_2^k$ ,

Now let  $t_2 \in \text{Aut}(F_2)$ . Then  $t_2(x_1) = w_1(x_1, x_2)$  and  $t_2(x_2) = w_2(x_1, x_2)$ . Since  $t_1 t_2 = t_2 t_1$ , we have that

$$w_1(x_1^{-1}, x_2) = w_1(x_1, x_2)^{-1}, \quad w_2(x_1^{-1}, x_2) = w_2(x_1, x_2).$$

As before, the second relation implies that the word  $w_2 = x_2^k$ ,  $k \in \mathbb{Z}$ . Looking at the first relation, we get that  $w_1 = c x_1^\ell c^{-1}$ ,  $\ell \in \mathbb{Z}$ . But  $w_1$  and  $w_2$  must be a generating set for  $F_2$ . That means that  $k, \ell \in \{\pm 1\}$ . Also,

$$\begin{aligned} (t_1 \circ t_2)(x_1) &= t_1(c(x_1, x_2) x_1^{\pm 1} c(x_1, x_2)^{-1}) = c(x_1^{-1}, x_2) x_1^{\mp 1} c(x_1^{-1}, x_2)^{-1} \\ (t_2 \circ t_1)(x_1) &= t_2(x_1^{-1}) = c(x_1, x_2) x_1^{\mp 1} c(x_1, x_2)^{-1} \end{aligned}$$

Since  $t_1 \circ t_2 = t_2 \circ t_1$ ,  $c(x_1, x_2) = c(x_1^{-1}, x_2)$  and thus  $c = x_2^k$ ,  $k \in \mathbb{Z}$ .  $\square$

**Lemma 5.2.** *Let  $G = F_2 \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$  where the generator  $t_1$  of  $\mathbb{Z}/2\mathbb{Z}$  acts as*

$$t_1(x_1) = x_1^{-1}, \quad t_1(x_2) = x_2$$

*and the generator  $t_2$  of  $\mathbb{Z}$  acts as*

$$t_2(x_1) = x_2^k x_1^{\pm 1} x_2^{-k}, \quad t_2(x_2) = x_2^{\pm 1}.$$

*Then  $G$  is a CAT(0)-group.*

*Proof.* The group  $G$  has the following presentation:

$$\langle t_1, t_2, x_1, x_2 : t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, t_1^2 = [t_1, t_2] = 1, t_2 x_1 t_2^{-1} = x_2^k x_1^{\pm 1} x_2^{-k}, t_2 x_2 t_2^{-1} = x_2^{\pm 1} \rangle.$$

We set  $\xi = x_2^{-k} t_2$ . Then the presentation becomes:

$$\langle t_1, x_1, x_2, \xi : t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, t_1^2 = [t_1, \xi] = 1, \xi x_1 \xi^{-1} = x_1^{\pm 1}, \xi x_2 \xi^{-1} = x_2^{\pm 1} \rangle.$$

Now we consider four cases:

Case 1. In this case,

$$G = \langle t_1, x_1, x_2, \xi : t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, t_1^2 = [t_1, \xi] = 1, \xi x_1 \xi^{-1} = x_1, \xi x_2 \xi^{-1} = x_2 \rangle.$$

Now, let  $L_1 = \langle x_2, t_1, \xi : t_1^2 = [t_1, x_2] = [t_1, \xi] = [x_2, \xi] = 1 \rangle < G$  which is isomorphic to  $\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$  and thus it is CAT(0). Also, let  $L_2 = \langle x_1, t_1, \xi : t_1 x_1 t_1^{-1} = x_1^{-1}, \xi x_1 \xi^{-1} = x_1, t_1^2 = [t_1, \xi] = 1 \rangle$ . Then

$$L_2 = \langle \xi \rangle \times \langle x_1, t_1 : t_1 x_1 t_1^{-1} = x_1^{-1} \rangle \cong \mathbb{Z} \times D_\infty.$$

The infinite dihedral group is CAT(0)-group because it is a Coxeter group ([13]). Thus  $L_2$  is a CAT(0)-group, as a direct product of CAT(0)-groups. Also,  $L = \langle t_1, \xi : t_1^2 = [t_1, \xi] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . But then  $G = L_1 *_L L_2$  is CAT(0) ([5], Part II, Corollary 11.19).

Case 2. We assume,

$$G = \langle t_1, x_1, x_2, \xi : t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, t_1^2 = [t_1, \xi] = 1, \xi x_1 \xi^{-1} = x_1^{-1}, \xi x_2 \xi^{-1} = x_2 \rangle.$$

By setting  $\xi_2 = t_1 \xi$  and rewriting the presentation we are back in Case I.

Case 3. We assume,

$$G = \langle t_1, x_1, x_2, \xi : t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, t_1^2 = [t_1, \xi] = 1, \xi x_1 \xi^{-1} = x_1, \xi x_2 \xi^{-1} = x_2^{-1} \rangle.$$

We repeat the same method as before. Let

$$L_1 = \langle x_2, t_1, \xi : t_1^2 = [t_1, x_2] = [t_1, \xi] = 1, \xi x_2 \xi^{-1} = x_2^{-1} \rangle \cong \langle t_1 \mid t_1^2 = 1 \rangle \times \langle x_2, \xi : \xi x_2 \xi^{-1} = x_2^{-1} \rangle$$

Therefore  $L_1 \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z} \rtimes \mathbb{Z})$ . Now the second group can be written as an HNN-extension  $\mathbb{Z} *_r$ , where  $r$  is the non-trivial automorphism of  $\mathbb{Z}$ . Then  $\mathbb{Z} \rtimes \mathbb{Z}$  is CAT(0)-group ([5], Part II, Corollary 11.22), and thus  $L_1$  is CAT(0). Now  $L$  and  $L_2$  are as in Case 1, and thus  $G = L_1 *_L L_2$  is CAT(0).

Case 4. We assume,

$$G = \langle t_1, x_1, x_2, \xi : t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, t_1^2 = [t_1, \xi] = 1, \xi x_1 \xi^{-1} = x_1^{-1}, \xi x_2 \xi^{-1} = x_2^{-1} \rangle.$$

Again set  $\xi_2 = t_1 \xi$  and rewrite the presentation to arrive at Case 3.

□

**Proposition 5.3.** *The group  $F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) < \mathcal{H}_{(n-1)}$  is a CAT(0)-group.*

*Proof.* Let  $t_1$  be the generator of  $\mathbb{Z}/2\mathbb{Z}$  and  $t_2$  the generator of  $\mathbb{Z}$ . We will consider two cases:

Case 1. Let  $t_2 \in \text{Hol}(F_2)$  and  $t_2 \notin \text{Aut}(F_2)$ . From Lemma 5.1, Part (1),  $t_2$  is an element  $x_2^k$ ,  $k \in \mathbb{Z}$ ,  $k \neq 0$ . Then  $G = F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$  has the following presentation:

$$\langle x_1, x_2, \dots, x_n, t_1, t_2 : \begin{aligned} t_1^2 &= 1, t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_i t_1^{-1} = x_i, i = 2, \dots, n, \\ t_2 x_1 t_2^{-1} &= x_1, t_2 x_2 t_2^{-1} = x_2, t_2 x_i t_2^{-1} = x_2^k x_i x_2^{-k}, i = 3, \dots, n, [t_1, t_2] = 1 \end{aligned} \rangle$$

We change the generators by setting  $\xi = x_2^{-k} t_2$ . First notice that  $[t_1, \xi] = 1$  because  $t_1$  commutes with  $t_2$  and  $x_2$ . Then the presentation becomes:

$$\langle x_1, x_2, \dots, x_n, t_1, \xi : \begin{aligned} t_1^2 &= 1, t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_i t_1^{-1} = x_i, i = 2, \dots, n, \\ \xi x_1 \xi^{-1} &= x_2^{-k} x_1 x_2^k, \xi x_i \xi^{-1} = x_i, i = 2, \dots, n, [t_1, \xi] = 1 \end{aligned} \rangle$$

Set  $K_1 = \langle t_1, \xi, x_3, \dots, x_n : t_1^2 = [t_1, \xi] = [t_1, x_i] = [\xi, x_i] = 1, i = 3, \dots, n \rangle < F_n \rtimes (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$  and it is isomorphic to  $\mathbb{Z}^{n-1} \times \mathbb{Z}/2\mathbb{Z}$ , which is a CAT(0)-group. Let  $K = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle t_1, t_2 \rangle$ , which is a virtually infinite cyclic. Also, set  $K_2 < G$  with presentation:

$$\langle x_1, x_2, t_1, \xi : t_1^2 = 1, t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, \xi x_1 \xi^{-1} = x_2^{-k} x_1 x_2^k, \xi x_2 \xi^{-1} = x_2, [t_1, \xi] = 1 \rangle.$$

Notice that  $G = K_1 *_K K_2$ . In order to show that  $G$  is CAT(0), it suffices to show that  $K_2$  is a CAT(0) group.

To that end, we change generators in  $K_2$  by setting  $\zeta = x_2^k \xi$ . Then the presentation of  $K_2$  becomes:

$$\langle x_1, x_2, t_1, \zeta : t_1^2 = 1, t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, \zeta x_1 \zeta^{-1} = x_1, \zeta x_2 \zeta^{-1} = x_2, [t_1, \zeta] = 1 \rangle.$$

For Lemma 5.2, Case 1,  $K_2$  is CAT(0) and we are done.

Case 2. We assume  $t_2 \in \text{Aut}(F_2)$ . Using Lemma 5.2, Part (2), the group  $G$  has presentation:

$$\langle x_1, x_2, \dots, x_n, t_1, t_2 : \begin{aligned} t_1^2 &= 1, t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_i t_1^{-1} = x_i, i = 2, \dots, n, \\ t_2 x_1 t_2^{-1} &= x_2^k x_1^{\pm 1} x_2^{-k}, t_2 x_2 t_2^{-1} = x_2^{\pm 1}, t_2 x_i t_2^{-1} = x_i, i = 3, \dots, n, [t_1, t_2] = 1 \end{aligned} \rangle$$

Set  $K_1 = \langle t_1, t_2, x_3, \dots, x_n : t_1^2 = [t_1, t_2] = [t_1, x_i] = [t_2, x_i] = 1, i = 3, \dots, n \rangle \cong \mathbb{Z}^{n-1} \times \mathbb{Z}/2\mathbb{Z}$  and  $K = \langle t_1, t_2 \rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  which are two subgroups of  $G$ . Finally, set  $K_2$  to be

$$\langle t_1, t_2, x_1, x_2 : t_1^2 = 1, t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, t_2 x_1 t_2^{-1} = x_2^k x_1^{\pm 1} x_2^{-k}, t_2 x_2 t_2^{-1} = x_2^{\pm 1}, [t_1, t_2] = 1 \rangle.$$

It is obvious that  $G = K_1 *_K K_2$ . To show that  $G$  is CAT(0) it suffices to show that  $K_2$  is a CAT(0) groups.

Set  $\zeta = x_2^{-k} t_2$  and the presentation becomes:

$$\langle t_1, x_1, x_2, \zeta : t_1^2 = 1, t_1 x_1 t_1^{-1} = x_1^{-1}, t_1 x_2 t_1^{-1} = x_2, \zeta x_1 \zeta^{-1} = x_1^{\pm 1}, \zeta x_2 \zeta^{-1} = x_2^{\pm 1}, [t_1, \zeta] = 1 \rangle,$$

which is CAT(0) from Lemma 5.2. □

The reader should notice that there are more possibilities for the  $\mathbb{Z}/2\mathbb{Z}$  action on the  $F_2 \rtimes \mathbb{Z}$  subgroups of  $\text{Aut}(F_2)$ .

## REFERENCES

- [1] Bartels, Arthur C. *On the domain of the assembly map in algebraic K-theory*. *Algebr. Geom. Topol.* **3**, 1037-1050 (2003).
- [2] Bartels, Arthur; Reich, Holger. *Coefficients for the Farrell-Jones conjecture*. *Adv. Math.* **209**, No. 1, 337-362 (2007).
- [3] Bartels, Arthur; Lück, Wolfgang; Reich, Holger. *The K-theoretic Farrell-Jones conjecture for hyperbolic groups*. *Invent. Math.* **172**, No. 1, 29-70 (2008).
- [4] Brady, Thomas. *Complexes of nonpositive curvature for extensions of  $F_2$  by  $\mathbb{Z}$* . *Topology Appl.* **63**, No.3, 267-275 (1995).
- [5] Bridson, Martin; Haefliger, André. *Metric spaces of non-positive curvature*, Volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.
- [6] Davis, James F.; Quinn, Frank; Reich, Holger. *Algebraic K-theory over the infinite dihedral group: a controlled topology approach*. *J. Topol.* **4**, No. 3, 505-528 (2011).
- [7] Davis, Michael W.; Januszkiewicz, Tadeusz. *Right-angled Artin groups are commensurable with right-angled Coxeter groups*. *J. Pure Appl. Algebra* **153**, No.3, 229-235 (2000).
- [8] Formanek, Edward; Procesi, Claudio. *The automorphism group of a free group is not linear*. *J. Algebra* **149** (1992), no. 2, 494-499.
- [9] Lück, Wolfgang; Weiermann, Michael. *On the classifying space of the family of virtually cyclic subgroups*. *Pure Appl. Math. Q.* **8**, No. 2, 497-555 (2012).
- [10] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney 1966.
- [11] S. Meskin, *Periodic automorphisms of the two-generator free group*, Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973), pp. 494-498. *Lecture Notes in Math.*, Vol. 372, Springer, Berlin, 1974.
- [12] Metaftsis, Vassilis; Prassidis, Stratos. *On the vanishing of the lower K-theory of the holomorph of a free group on two generators*. *Forum Math.* **24** (2012), no. 4, 861-874.
- [13] Moussong, Gábor. *Hyperbolic Coxeter groups*, Ph.D Thesis, The Ohio State University, 1988.
- [14] Roushon, S. K. *Algebraic K-theory of groups wreath product with finite groups*. *Topology Appl.* **154** (2007), no. 9, 1921-1930.
- [15] Samuelson, Peter. *On  $CAT(0)$  structures for free-by-cyclic groups*. *Topology Appl.* **153** (2006), no. 15, 2823-2833.
- [16] Wegner, Christian. *The Farrell-Jones Conjecture for virtually solvable groups*. arXiv:1308.2432 [math.GT].
- [17] Wegner, Christian. *The K-theoretic Farrell-Jones conjecture for  $CAT(0)$ -groups*. *Proc. Am. Math. Soc.* **140**, No. 3, 779-793 (2012).
- [18] Weibel, Charles.  *$NK_0$  and  $NK_1$  of the groups  $C_4$  and  $D_4$* . *Comment. Math. Helv.* **84**, No. 2, 339-349 (2009).

DEPARTMENT OF MATHEMATICS UNIVERSITY OF THE AEGEAN, KARLOVASSI, SAMOS, 83200 GREECE  
*E-mail address:* vmet@aegean.gr

DEPARTMENT OF MATHEMATICS UNIVERSITY OF THE AEGEAN, KARLOVASSI, SAMOS, 83200 GREECE  
*E-mail address:* prasside@aegean.gr